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Off-Diagonal Long-Range Order and the Meissner Effect

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We provide a general, model-independent, derivation of the Meissner effect, on the basis of assumptions of off-diagonal long-range order (ODLRO) and gauge covariance of the second kind. This is an exact result that is independent of the microscopic mechanism responsible for the ordering, and so is applicable both to high- and low-T_c superconductors.

KEY WORDS: Gauge covariance; ODLRO; Meissner Effect.

1. INTRODUCTION

This note is devoted to a treatment of an old problem in the theory of superconductivity, which, in our view, has not been satisfactorily resolved either for metallic or for high-temperature superconductors. The problem is that of the microscopic derivation of the Meissner effect, which is, of course, the key to the electromagnetic properties of superconductors.[1]

In the theory of metallic superconductivity, there is the radical difficulty that the Bardeen–Cooper–Schrieffer (BCS) ansatz[2] violates the principle of gauge covariance of the second kind.[3] Attempts[4] to overcome this difficulty, by taking account of interactions other than those of the Cooper pairing, have led to derivations of the Meissner effect that are only approximately gauge invariant. Since exact gauge covariance is required for local charge conservation and thus even for the very definition of electric current, this is an unsatisfactory state of affairs. As regards high-T_c superconductivity, developments toward a microscopic theory[5] have not yet led to a clear-cut derivation of the Meissner effect.

The object of the present note is to provide a simple and exact derivation of the Meissner effect from very general principles that do not require

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the explicit form of the microstate of a superconductor. To this end, we invoke the assumption of off-diagonal long-range order (ODLRO), which was proposed long ago as a characterization of superfluidity in both liquid helium[6] and in superconductors[7]: this assumption is satisfied by the ansätze of Feynman[8] for helium, of BCS for metallic superconductors, and of some proposed theories[15] of high-T_c ones.

The result we shall obtain, by an exact symmetry argument, is that the Meissner effect is a general consequence of ODLRO and gauge covariance of the second kind: specifically, we prove that this covariance renders ODLRO incompatible with the entry of a uniform magnetic field into the system. It is noteworthy that the gauge principle is an essential factor in our derivation of this result, whereas it appeared to be an obstacle to previous theories. The key to our result is that, as we shall see in Section 2, space translations in the presence of a uniform magnetic field B are represented by transformations of the quantized field ψ of the form[9]

$$\psi(x) \to \psi(x+a) \exp\left(\frac{-ie(B, a, x)}{2\hbar c}\right) \tag{1.1}$$

The exponential factor arises from the regauging of the magnetic vector potential corresponding to a spatial displacement a. It is this factor that renders ODLRO incompatible with the presence of the magnetic field.

We organize our material as folows. In Section 2, we formulate the general model for the theory. In Section 3, we reformulate the ODLRO condition and state our main result in the form of a proposition and subsequent discussion. In Section 4, we prove the proposition.

For expository reasons, we keep the mathematics very simple, using the standard second quantization formalism of condensed matter physics. We remark, however, that our argument can easily be put onto a rigorous basis within the framework of algebraic statistical mechanics.

2. THE GENERAL MODEL

Our model Σ is an infinitely extended system of particles in a space X, which may be either a Euclidean continuum or a lattice. Points in X will generally be denoted by x, sometimes by y, a, or b. It will be assumed that Σ consists of a system Σ_0 of charged particles of one species in X and possibly of another component Σ_1. It will be assumed that the model enjoys the properties of gauge covariance of both the first and the second kind, and that its interactions are translationally invariant. These assumptions are satisfied by both Fröhlich's electron–phonon model[10] and

Hubbard's strong repulsion model,[11] on which theories of metallic and high-T_c superconductivity, respectively, are based.

We shall describe the subsystem Σ_0 in terms of a quantized field $\psi = (\psi_\uparrow, \psi_\downarrow)$, which satisfies the canonical commutation or anticommutation relations, the former alternative admitting the possibility that Σ_0 might be a Bose system and so covering models of high-T_c superconductivity where the charge carriers are bosons, e.g., "holons."[5] The observables of Σ_0 are formed from the polynomials in ψ and ψ^* that are invariant under gauge transformations of the first kind, i.e., $\psi \to \psi e^{i\alpha}$, with α constant. Thus, they are generated algebraically from operators of the form $\psi^*(x_1) \cdots \psi^*(x_n)\, \psi(x_{n+1}) \cdots \psi(x_{2n})$. We shall denote by $Q(t)$ the time-translate, in Heisenberg representation, of an arbitrary observable Q. A dynamical characterization of thermal equilibrium states of Σ at inverse temperature β is then given by the Kubo–Martin–Schwinger condition, which is valid even for infinite systems,[12] i.e.,

$$\langle Q_1(t) Q_2 \rangle = \langle Q_2 Q_1(t + i\hbar\beta) \rangle \tag{2.1}$$

for arbitrary observables Q_1, Q_1, where the angular brackets denote expectation value. In view of the assumed translational invariance of the interactions, it follows from this formula that Σ supports translationally invariant equilibrium states.

We shall be concerned with the properties of the system in the presence of a classical magnetic field $B = \operatorname{curl} A$, and we shall assume that its dynamics is covariant w.r.t. gauge transformations of the second kind,

$$A(x) \to A(x) + \nabla\phi(x), \qquad \psi(x) \to \psi(x) \exp\left(\frac{ie\phi(x)}{\hbar c}\right) \tag{2.2}$$

where ϕ is an arbitrary function of position and $-e$ is the electronic charge. We shall also assume that the interparticle interactions of the model are invariant under space translations, i.e., that the dynamics is covariant w.r.t.

$$A(x) \to A(x + a), \qquad \psi(x) \to \psi(x + a) \tag{2.3}$$

together with the transformation of Σ_1 observables corresponding to the spatial displacement a. Specializing now to the case where the magnetic induction B is uniform and so may be represented by the vector potential $A(x) = \frac{1}{2} B \times x$ and choosing $\phi(x) = -(B, x, a)$, we have the relation $A(x) + \nabla\phi(x) = A(x - a)$. Hence, it follows from (2.2) and (2.3) that the dynamics of Σ is covariant w.r.t.

$$\psi(x) \to \gamma(a)\, \psi(x) \equiv \psi(x + a) \exp\left(\frac{-ie(B, x, a)}{2\hbar c}\right) \tag{2.4}$$

together with the transformation for the Σ_1 observables, corresponding to the space translation $x \to x + a$. Thus, the transformations γ correspond to space translations of the Σ_0 observables.

3. ODLRO AND THE MEISSNER EFFECT

In order to formulate ODLRO, we introduce the pair field

$$\Psi(x_1, x_2) = \psi_\uparrow(x_1)\, \psi_\downarrow(x_2) \tag{3.1}$$

The property of ODLRO may then be expressed in terms of this field by the condition that[7]

$$\lim_{|y| \to \infty} [\langle \Psi^*(x_1 + y, x_2 + y)\, \Psi(x'_1, x'_2)\rangle$$
$$- \Phi^*(x_1 + y, x_2 + y)\, \Phi(x'_1, x'_2)] = 0, \qquad \forall x'_1, x'_2, x_1, x_2 \in X \tag{3.2}$$

where the function Φ is c-number-valued, and $\Phi(x_1 + y,\ x_2 + y)$ does not tend to zero as $|y| \to \infty$. We note here that Ψ is not an observable, though the quantity in $\langle \cdot \rangle$ in (3.2) is one.

The following proposition, which we shall prove in Section 4, and the subsequent comments constitute our derivation of the Meissner effect.

Proposition. Assume that the system is in a translationally invariant state possessing the property of ODLRO. Then:

(a) In the case where X is a continuum, the system cannot support a (nonzero) uniform magnetic field.

(b) In the case where X is a lattice, whose cells are of volume v, the internal, uniform magnetic field B is restricted to values $\hbar cx/ev$, with x a point of the lattice X.

(c) In the latter case and under the further assumption that B is proportional to the applied magnetic field H, the system cannot support a (nonzero) magnetic field.

Comments. (1) This Proposition implies that, under the stated conditions, translationally invariant (including equilibrium) states with ODLRO do not admit uniform magnetic fields, and thus that they exhibit the Meissner effect: by contrast, states that are normally diamagnetic admit such fields.

(2) In order to relate the Proposition to superconductivity, as characterized by the Meissner effect,[1] we now explicitly assume that Σ undergoes a phase transition at some temperature T_c such that, for $T < T_c$

only, the equilibrium state has the ODLRO property and that this property persists even in the presence of a sufficiently weak uniform magnetic field. Under this assumption, the Proposition tells us that the ODLRO phase is superconducting.

(3) Since one knows empirically[(3)] that type II superconductors take up spatially nonuniform structures when the strength of the applied magnetic field H is less than a critical value H_{c_1}, the dependence of the proposition on the translational invariance of the state limits[2] its applicability to these materials to situations where $H < H_{c_1}$.

(4) The result of part (b) of the Proposition, without the hypothesis of (c), tells us that the only alternative to the Meissner effect for a lattice system is the admission of a field B of magnitude $\sim \hbar c/el^2$, where l is the lattice spacing. For typical values of l, e.g., $\sim 10^{-8}$, this would mean that the internal field would be of the order of $\sim 10^9$ G, which is many orders of magnitude larger than any known critical fields. This lends empirical support for the result in (c) for lattice systems, obtained on the basis of the supplementary assumption there.

4. PROOF OF PROPOSITION

By Eqs. (2.4) and (3.1),

$$\begin{aligned}\gamma(a)\,\Psi(x_1, x_2) &\equiv \Psi_a(x_1, x_2)\\ &= \Psi(x_1 + a, x_2 + a)\exp\left(\frac{-ie(B, x_1 + x_2, a)}{2\hbar c}\right)\end{aligned} \tag{4.1}$$

In analogy with this formula, we define $g(a)$ to the transformation of complex-valued two-point functions by the equation

$$\begin{aligned}g(a)\,\Phi(x_1, x_2) &\equiv \Phi_a(x_1, x_2)\\ &= \Phi(x_1 + a, x_2 + a)\exp\left(\frac{-ie(B, x_1 + x_2, a)}{2\hbar c}\right)\end{aligned} \tag{4.2}$$

and thence infer that

$$g(-a-b)[g(a)\,g(b) - g(b)\,g(a)]\,\Phi = 2i\Phi \sin\left(\frac{e(B, a, b)}{\hbar c}\right) \tag{4.3}$$

[2] I am grateful to Oliver Penrose for drawing my attention to this point.

By (4.1) and (4.2), the ODLRO condition (3.2) is invariant under $\Psi \to \Psi_a$, $\Phi \to \Phi_a$, and thus

$$\lim_{|y| \to \infty} (\langle \Psi_a^*(x_1+y, x_2+y)\, \Psi_a(x_1', x_2')\rangle - \Phi_a^*(x_1+y, x_2+y)\, \Phi_a(x_1', x_2')) = 0 \tag{4.4}$$

Further, by the translational invariance assumption for the state,

$$\langle \Psi_a^*(x_1+y, x_2+y)\, \Psi_a(x_1', x_2')\rangle \equiv \langle \Psi^*(x_1+y, x_2+y)\, \Psi(x_1', x_2')\rangle$$

and therefore (4.4) may be rewritten as

$$\lim_{|y| \to \infty} [\langle \Psi^*(x_1+y, x_2+y)\, \Psi(x_1', x_x')\rangle - \Phi_a^*(x_1+y, x_2+y)\, \Phi_a(x_1', x_2')] = 0$$

On subtracting this formula from (3.2), we see that

$$\lim_{|y| \to \infty} [\Phi_a^*(x_1+y, x_2+y)\, \Phi_a(x_1', x_2') - \Phi^*(x_1+y, x_2+y)\, \Phi(x_1', x_2')] = 0 \tag{4.5}$$

Since this remains valid when we replace x_1', x_2' by x_1, x_2,

$$\lim_{|y| \to \infty} [\Phi_a^*(x_1+y, x_2+y)\, \Phi_a(x_1, x_2) - \Phi^*(x_1+y, x_2+y)\, \Phi(x_1, x_2)] = 0 \tag{4.6}$$

On multiplying (4.5) by $\Phi(x_1, x_2)$ and (4.6) by $\Phi(x_1', x_2')$ and subtracting, we see that

$$\lim_{|y| \to \infty} \{\Phi^*(x_1+y, x_2+y)[\Phi_a(x_1, x_2)\, \Phi(x_1', x_2') - \Phi(x_1, x_2)\, \Phi_a(x_1', x_2')]\} = 0$$

Therefore, since, by our above definition of ODLRO, $\Phi(x_1+y, x_2+y)$ does not tend to zero as $|y| \to \infty$,

$$\Phi_a(x_1, x_2)\, \Phi(x_1', x_2') = \Phi(x_1, x_2)\, \Phi_a(x_1', x_2') \qquad \forall x_1, x_2, x_1', x_2' \in X \tag{4.7}$$

A further simple consequence of the condition that $\Phi(x_1+y, x_2+y)$ does not tend to zero as $|y| \to \infty$ is that Φ does not vanish everywhere. Thus, choosing $(\bar{x}_1, \bar{x}_2)$ to be a point pair where this function is nonzero and defining $\lambda(a)$ to be the ratio of $\Phi_a(\bar{x}_1, \bar{x}_2)$ to $\Phi(\bar{x}_1, \bar{x}_2)$, we see from (4.7) that $\lambda(a)$ is independent of the choice of $(\bar{x}_1, \bar{x}_2)$, i.e., that $\Phi_a = g(a)\Phi = \lambda(a)\Phi$. Hence,

$$g(a)\, g(b)\Phi = g(b)\, g(a)\Phi \qquad \forall a, b \in X$$

since the c-numbers $\lambda(a)$, $\lambda(b)$ intercommute; and consequently, by (4.3),

$$\Phi \sin\left(\frac{e(B, a, b)}{\hbar c}\right) = 0$$

Consequently, as the function Φ is not identically zero, the sinusoidal term in this equation must vanish, and therefore

$$(B, a, b) = \pi n \hbar c/e \tag{4.8}$$

where n is an integer that depends on a and b.

In the case where X is a continuum, this formula implies that $B = 0$, as may be seen on replacing a by ka with k an irrational number. This proves part (a) of the proposition.

In the case where X is a lattice, with basis vectors a_1, a_2, a_3, it follows from (4.8) that

$$(B, a_i, a_j) = \pi n_{ij} \hbar c/e \tag{4.9}$$

where n_{ij} is an integer. Expressing B as a linear combination of a_1, a_2, a_3, we see from (4.9) that the coefficient of a_i is $\pi v_i \hbar c/ev$, where v_i is an integer and v is the volume of a lattice cell, (a_1, a_2, a_3). Hence, B is of the form $\hbar c x/ev$, where $x = \sum_i v_i a_i$ is an arbitrary lattice point, as required for part (b) of the Proposition. Moreover, if B is proportional to the applied field H, it follows immediately from this result that $B = 0$. This completes the proof of the Proposition.

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